SYNOPSIS

Ph.D in Mathematics

A DESCRIPTIVE STUDY ON SECOND ORDER BIPOLAR FUZZY STRUCTURES

Submitted by

Muthamizhselvi. S

(19PHMAF004)

Supervisor

Dr. V.M. Vijayalakshmi

Assistant Professor,

Department of Science and Humanities

Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women,

Coimbatore-641043.

October 2024

A DESCRIPTIVE STUDY ON SECOND ORDER BIPOLAR FUZZY STRUCTURES

Introduction

Decision making is one of the major techniques which provides a schematic representation of human reasoning in every real world complexities. To handle the uncertainties in the problem, Zadeh (1965) proposed the concept of fuzzy set theory. Fuzzy sets have been used to generalise many concepts related to the problems of decision making.

In general, human decision making involves two-sided information that is Bipolarity. Zhang (1994) introduced the concept of bipolar fuzzy set by combining the polarity and fuzziness into a unified structure.

In this thesis the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously. Similarly 'bipolar fuzzy set' and 'first order bipolar fuzzy set' are used synonymously. Similar terminology applies to all concepts related to first order fuzzy sets and first order bipolar fuzzy sets.

In 1975, Zadeh proposed the concept of second order fuzzy set which is defined as a map from the set X to I^I where I is the unit interval [0,1].

The topological structures of set theories dealing with uncertainties were studied by many eminent mathematicians. The concept of fuzzy topology was first introduced by Chang in 1968 and then modified by Lowen (1976).

Using second order fuzzy sets, Kalaichelvi (2007) extended first order fuzzy topology of Chang and Lowen to second order fuzzy topology.

The aim of this thesis is to construct second order bipolar fuzzy structures by combining first order bipolar fuzzy structures and second order fuzzy structures.

In this study the concepts 'second order bipolar fuzzy set', 'second order bipolar fuzzy topology' and 'second order bipolar fuzzy matrix' are introduced.

Objectives:

- To introduce the notion of second order bipolar fuzzy set, second order bipolar fuzzy topology and second order bipolar fuzzy matrix.

- To obtain the relations between first and second order bipolar fuzzy topological spaces and to construct the relations between second order bipolar fuzzy topological spaces and crisp topological spaces.
- To introduce the concepts of product, separation axioms, gradation of openness in First Order Bipolar Fuzzy Topological Spaces (FOBPFTS).
- To study the concepts of second order bipolar fuzzy structures such as product, continuity, separation axioms, gradation of openness and compactness.
- To obtain the solution for the decision making problem using second order bipolar fuzzy TOPSIS method.

Methodology:

The research methods carried out in the present study are as follows:

- Conceptual research method to define SOBPFS, SOBPFT and its structures.
- Analytical method for proving results.
- Relational analysis between FOBPFTS and SOBPFTS.
- Examples and counter examples for implications and nonimplications.

In the present work, the concepts of second order bipolar fuzzy set and second order bipolar fuzzy topology are introduced and various second order bipolar fuzzy structures are studied.

The concepts studied are

(i) Second order bipolar fuzzy continuity
(ii) Second order bipolar fuzzy product topology
(iii) Different versions of Hausdorff separation axioms in second order bipolar fuzzy topological space.
(iv) Second order bipolar fuzzy gradation of openness.
(v) Second order bipolar fuzzy compactness

The content of the thesis is presented in six chapters.

Chapter 1:

Preliminary definitions such as

- First order fuzzy set
- First order fuzzy topological spaces

- Second order fuzzy set
- Second order fuzzy topological spaces
- First order bipolar fuzzy set
- First order bipolar fuzzy topological spaces
- First order fuzzy matrix
- First order bipolar fuzzy matrix

are presented in the first chapter.

Chapter 2:

In the first section, definitions and properties of second order bipolar fuzzy sets and second order bipolar fuzzy topological spaces are introduced and studied.

"**Second order bipolar fuzzy set:** A second order bipolar fuzzy set (SOBPFS) in X is a pair $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)$ where $\widehat{A}_{bp}{}^{+}: X \to [0,1]^{[0,1]}$ such that $\widehat{A}_{bp}{}^{+}(x)(\alpha) \in [0,1]$ and $\widehat{A}_{bp}{}^{-}: X \to [-1,0]^{[0,1]}$ such that $\widehat{A}_{bp}{}^{-}(x)(\alpha) \in [-1,0]$, for every $\alpha \in I$ and for every $x \in X$".

A second order bipolar fuzzy topology $\hat{\tau}_{\mathfrak{B}}$ is defined as a direct extension of first order bipolar fuzzy topology of Azhagappan and Kamaraj (2016) and Kim et al.., (2019).

In the second section, the relations between first order bipolar fuzzy and second order bipolar fuzzy topological spaces are analysed. Examples for second order bipolar fuzzy topological spaces are also given.

The Relations analysed are

(i) With every first order bipolar fuzzy topology $\tau_{\mathfrak{B}}$ on X, a second order bipolar fuzzy topology $\hat{\tau}_{\mathfrak{B}}$ on X is related. The relation $\tau_{\mathfrak{B}} \to \hat{\tau}_{\mathfrak{B}}$ is referred as R_1.

(ii) With every second order bipolar fuzzy topology $\hat{\tau}_{\mathfrak{B}}$ on X and for every $x \in X$, a first order bipolar fuzzy topology $(\hat{\tau}_{\mathfrak{B}})_x$ on I is related. The relation $\hat{\tau}_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_x$ is referred as R_2.

(iii) With every second order bipolar fuzzy topology $\tau_{\mathfrak{B}}$ on X and for every $\alpha \in I$, a first order bipolar fuzzy topology $(\hat{\tau}_{\mathfrak{B}})_\alpha$ on X is related. The relation $\tau_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_\alpha$ is referred as R_3.

(iv) With every first order bipolar fuzzy topology $\tau_{\mathfrak{B}}$ on I, a second order bipolar fuzzy topology $(\hat{\tau}_{\mathfrak{B}})_I$ on X is related. The relation $\tau_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_I$ is referred as R_4.

(v) The complement of second order bipolar fuzzy set $\widehat{A}_{bp}$ on X is defined in two different ways and they are denoted as $\left(\widehat{A}_{bp}\right)_c$ and $\left(\widehat{A}_{bp}\right)^c$.

If $\hat{\tau}_{\mathfrak{B}} = \{(\widehat{A}_{bp})_{\lambda}/(A_{bp})_{\lambda} \in \tau_{\mathfrak{B}}\}$ is a second order bipolar fuzzy topology on X, then $(\hat{\tau}_{\mathfrak{B}})_c = \left\{\left((\widehat{A}_{bp})_{\lambda}\right)_c/(\widehat{A}_{bp})_{\lambda} \in \hat{\tau}_{\mathfrak{B}}\right\}$ is a second order bipolar fuzzy topology on X. The relation $\tau_{\mathfrak{B}} \to (\hat{\tau}_{\mathfrak{B}})_c$ is referred as R_5.

In the third section, relations between crisp topology τ on X and the second order bipolar fuzzy topology on X are discussed. From a crisp topology τ on X, three different second order bipolar fuzzy topologies denoted by $\widehat{\omega(\tau)}$, $\widehat{\omega_*(\tau)}$ and $\widehat{\omega_\varepsilon(\tau)}$ are constructed. Similarly, from a second order bipolar fuzzy topology $\hat{\tau}_{\mathfrak{B}}$ on X, three crisp topologies denoted by $i(\hat{\tau}_{\mathfrak{B}})$, $i^*(\hat{\tau}_{\mathfrak{B}})$ and $i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$ are constructed. Some relations between these structures are also established.

The following propositions are proved:

1) Let τ_1, τ_2 be two topologies on X such that $\tau_1 \subseteq \tau_2$. Then

   i. $\widehat{\omega(\tau_1)} \subseteq \widehat{\omega(\tau_2)}$

   ii. $\widehat{\omega_\varepsilon(\tau_1)} \subseteq \widehat{\omega_\varepsilon(\tau_2)}$

   iii. $\widehat{\omega_*(\tau_1)} \subseteq \widehat{\omega_*(\tau_2)}$

2) Let (X, τ) be a topological space. Then

   i. $\tau \subseteq i^*\left(\widehat{\omega_*(\tau)}\right)$

   ii. For $\varepsilon \in (0,1)$, $\tau \subseteq i_\varepsilon\left(\widehat{\omega_\varepsilon(\tau)}\right)$

   iii. $\tau \subseteq i\left(\widehat{\omega(\tau)}\right)$

Chapter 3:

Third chapter deals with second order bipolar fuzzy continuity and second order bipolar fuzzy product topology.

In the first section, second order bipolar fuzzy continuity is defined.

"**Second order bipolar fuzzy continuity:** A function $\theta: \left(X, \hat{\tau}_{\mathfrak{B}_1}\right) \to \left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is said to be second order bipolar fuzzy continuous, if the following condition is satisfied:

$\theta^{-1}\left(\widehat{A}_{bp}\right) \in \hat{\tau}_{\mathfrak{B}_1}$, if $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$."

It is proved that the relations R_1, R_3 and R_5 preserve continuity. It is also proved that the relations $\hat{\tau}_{\mathfrak{B}} \to i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$, $\hat{\tau}_{\mathfrak{B}} \to i^*(\hat{\tau}_{\mathfrak{B}})$, $\hat{\tau}_{\mathfrak{B}} \to i(\hat{\tau}_{\mathfrak{B}})$, $\tau \to \widehat{\omega_\varepsilon(\tau)}$, $\tau \to \widehat{\omega_*(\tau)}$ and $\tau \to \widehat{\omega(\tau)}$ are functorial.

In the second section, first order bipolar fuzzy product topology is introduced and its properties are analysed.

"**First order bipolar fuzzy product:** The product of first order bipolar fuzzy set $A_{bp} \times B_{bp}$ on $X \times Y$ is defined as follows: $A_{bp} \times B_{bp} = \left({A_{bp}}^{+} \times {B_{bp}}^{+}, {A_{bp}}^{-} \times {B_{bp}}^{-}\right)$ where

$\left({A_{bp}}^{+} \times {B_{bp}}^{+}\right)(x, y) = \min\{{A_{bp}}^{+}(x), {B_{bp}}^{+}(y)\}$ and

$\left({A_{bp}}^{-} \times {B_{bp}}^{-}\right)(x, y) = \max\{{A_{bp}}^{-}(x), {B_{bp}}^{-}(y)\}$, for every $(x, y) \in X \times Y$."

Some results are obtained.

In the third section, second order bipolar fuzzy product topology is defined.

Second order bipolar fuzzy product topology: Let $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$, $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be two second order bipolar fuzzy topological spaces. If $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$ and $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ then the **product** $\hat{A}_{bp} \times \hat{B}_{bp}$ on $X \times Y$ is defined as follows:

$\hat{A}_{bp} \times \hat{B}_{bp} = \left(\left({\hat{A}_{bp}}^{+} \times {\hat{B}_{bp}}^{+}\right), \left({\hat{A}_{bp}}^{-} \times {\hat{B}_{bp}}^{-}\right)\right)$ such that

$\left({\hat{A}_{bp}}^{+} \times {\hat{B}_{bp}}^{+}\right)(x, y)(\alpha) = {\hat{A}_{bp}}^{+}(x)(\alpha) \wedge {\hat{B}_{bp}}^{+}(y)(\alpha)$ and

$\left({\hat{A}_{bp}}^{-} \times {\hat{B}_{bp}}^{-}\right)(x, y)(\alpha) = {\hat{A}_{bp}}^{-}(x)(\alpha) \vee {\hat{B}_{bp}}^{-}(y)(\alpha)$,

for every $(x, y) \in X \times Y$ and for every $\alpha \in I$.

The **product topology** $\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$ on $X \times Y$ is the second order bipolar fuzzy topology having the collection $\{\hat{A}_{bp} \times \hat{B}_{bp} \,/\, \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}, \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}\}$ as a basis.

It is proved that the relations R_1, R_3 and R_5 preserve product.

Further the following propositions are obtained.

(i) $i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}_1}) \times i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}_2}) \subseteq i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$

(ii) $i^{*}\left(\hat{\tau}_{\mathfrak{B}_1}\right) \times i^{*}\left(\hat{\tau}_{\mathfrak{B}_2}\right) \subseteq i^{*}\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)$

(iii) $\widehat{\omega_{\varepsilon}(\tau)} \times \widehat{\omega_{\varepsilon}(\tau')} \subseteq \omega_{\varepsilon}\widehat{(\tau \times \tau')}$

(iv) $\widehat{\omega_{*}(\tau)} \times \widehat{\omega_{*}(\tau')} \subseteq \omega_{*}\widehat{(\tau \times \tau')}$

(v) $S_2\left(\hat{\tau}_{\mathfrak{B}_1}\right) \times S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right) \subseteq S_2\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)$

Chapter 4:

In chapter four, different versions of Hausdorff separation axioms in first order and second order bipolar fuzzy topological spaces are studied. Different versions of Hausdorff separation axioms have been defined and studied by many authors in the first order fuzzy topological spaces. Three different Hausdorff separation axioms in fuzzy topological spaces introduced by Gantner, Steinlage and Warren (1978), Katsaras (1981) and Srivastava et al. (1981) were extended to second order fuzzy topological spaces by Kalaichelvi (2011, 2013) and were denoted as W – Hausdorff, K – Hausdorff and S – Hausdorff axioms respectively.

In the first section, W – Hausdorff, K – Hausdorff and S – Hausdorff separation axioms of Warren, Katsaras and Srivastava are extended to first order bipolar fuzzy topological spaces.

"A first order bipolar fuzzy topological space $(X, \tau_{\mathfrak{B}})$ is said to be a first order bipolar fuzzy W-Hausdorff or bipolar fuzzy W-T_2, if for any two distinct points $x, y \in X$, there exists two bipolar fuzzy open sets $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ such that $A_{bp}{}^{+}(x) = 1, A_{bp}{}^{-}(x) = -1$, $B_{bp}{}^{+}(y) = 1, B_{bp}{}^{-}(y) = -1$ and $A_{bp} \cap B_{bp} = 0_{bp}$."

Propositions:

(i) Subspace of a bipolar fuzzy W-Hausdorff space is bipolar fuzzy W-Hausdorff space.

(ii) Product of two bipolar fuzzy W-Hausdorff spaces is a bipolar fuzzy W-Hausdorff space.

(iii) Arbitrary product of bipolar fuzzy W-Hausdorff spaces is bipolar fuzzy W-Hausdorff space.

These propositions are also proved for first order bipolar fuzzy K – Hausdorff and first order bipolar fuzzy S – Hausdorff spaces.

In the second section, W – Hausdorff, K – Hausdorff and S – Hausdorff separation axioms of Warren, Katsaras and Srivastava are extended to second order bipolar fuzzy topological spaces.

A second order bipolar fuzzy W – Hausdorff is defined in two different ways

"**Second order bipolar fuzzy W – Hausdorff space (Type 1):** A second order bipolar fuzzy topological space $(X, \hat{\tau}_{\mathfrak{B}})$ is said to be second order bipolar fuzzy W-Hausdorff space of type 1, denoted by $(SBPFW - H)_1$, if for every $x, y \in X$, $x \neq y$, there exists two bipolar fuzzy

open sets $\widehat{A}_{bp}, \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}}$ where $\widehat{A}_{bp} = \left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)$ and $\widehat{B}_{bp} = \left(\widehat{B}_{bp}{}^{+}, \widehat{B}_{bp}{}^{-}\right)$ such that $\widehat{A}_{bp}{}^{+}(x) = \mathbf{1}, \widehat{A}_{bp}{}^{-}(x) = -\mathbf{1}, \widehat{B}_{bp}{}^{+}(y) = \mathbf{1}, \widehat{B}_{bp}{}^{-}(y) = -\mathbf{1}$ & $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$."

"**Second order bipolar fuzzy W – Hausdorff space (Type 2):** second order bipolar fuzzy W-Hausdorff space of type 2, denoted by $(SBPFW - H)_2$, is defined by replacing the condition $\widehat{A}_{bp} \cap_1 \widehat{B}_{bp} = \hat{0}_{bp}$ in the above definition by $\widehat{A}_{bp} \cap_2 \widehat{B}_{bp} = \hat{0}_{bp}$."

In the same way second order bipolar fuzzy K - Hausdorff and second order bipolar fuzzy S – Hausdorff spaces are defined by following the definitions of Katsaras and Srivastava.

Propositions:

(i) Second order bipolar fuzzy subspace of $(SBPFW - H)_1$ space is $(SBPFW - H)_1$

(ii) Second order bipolar fuzzy subspace of $(SBPFW - H)_2$ space is $(SBPFW - H)_2$

(iii) Product of two $(SBPFW - H)_1$ spaces is $(SBPFW - H)_1$

(iv) Product of two $(SBPFW - H)_2$ spaces is $(SBPFW - H)_2$

(v) Arbitrary product of $(SBPFW - H)_1$ spaces is $(SBPFW - H)_1$

(vi) Arbitrary product of $(SBPFW - H)_2$ spaces is $(SBPFW - H)_2$

These propositions are also proved for second order bipolar fuzzy K – Hausdorff and second order bipolar fuzzy S – Hausdorff spaces.

Third section is devoted to study the relations between the first order bipolar fuzzy and second order bipolar fuzzy Hausdorff separation axioms with special reference to R_1, R_3 and R_5. The behaviour of these concepts with regards to $i(\hat{\tau}_{\mathfrak{B}})$, $i^*(\hat{\tau}_{\mathfrak{B}})$, $i_\varepsilon(\hat{\tau}_{\mathfrak{B}})$, $\widehat{\omega(\tau)}$, $\widehat{\omega_*(\tau)}$ and $\widehat{\omega_\varepsilon(\tau)}$ are also analysed.

Chapter 5:

The fifth chapter is devoted to the study of first order bipolar fuzzy gradation of openness, second order bipolar fuzzy gradation of openness and second order bipolar fuzzy compactness.

In 1992, Hazra et al. introduced the concept of gradation of openness and gave a new definition of fuzzy topology. Using this concept, Kalaichelvi (2000) introduced the definition

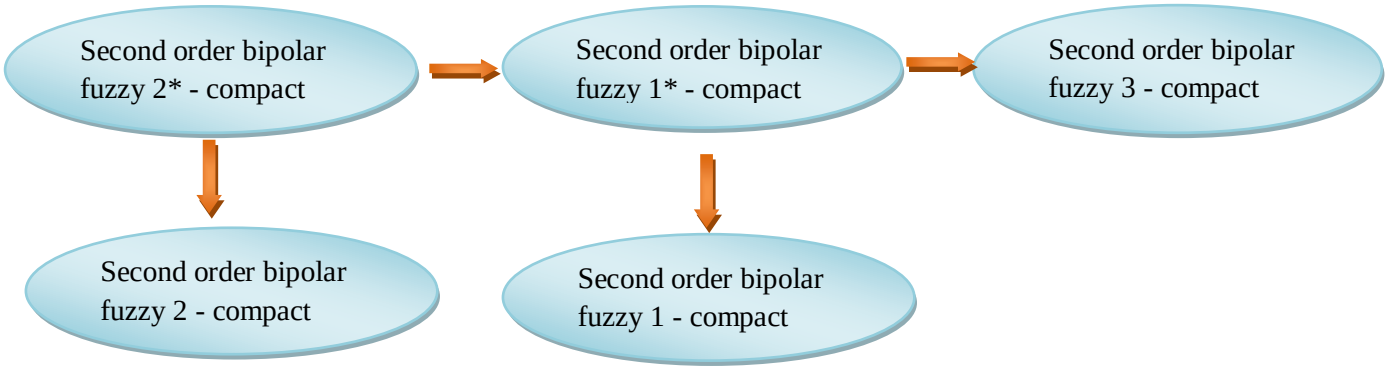
of second order gradation of openness and gave results related to the second order fuzzy topologies induced by second order gradation of openness.

In the first section, first order bipolar fuzzy gradation of openness is introduced and a new definition of first order bipolar fuzzy topology induced by first order bipolar fuzzy gradation of openness is given.

In the second section, second order bipolar fuzzy gradation of openness is introduced. Five interesting relations between first order bipolar fuzzy gradation of openness, second order bipolar fuzzy gradation of openness and first order gradation of openness are established. These relations are denoted as D_1, D_2, D_3, D_4 and D_5. It is proved that the relations D_1, D_3 and D_5 are functorial with respect to gradation preserving maps. Propositions related to the second order bipolar fuzzy topologies induced by the second order bipolar fuzzy gradation of openness are proved. Further it is proved that if $\hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}})$ is the second order bipolar fuzzy topology induced by the second order bipolar fuzzy gradation of openness $\hat{\mathcal{G}}$, then $\hat{\mathcal{G}} \to \hat{\tau}_{\mathfrak{B}}(\hat{\mathcal{G}})$ is functorial.

Third section is devoted to the study of compactness in the second order bipolar fuzzy topological spaces. Five different types of compactness, namely, second order bipolar fuzzy 1- compact, second order bipolar fuzzy 1*- compact, second order bipolar fuzzy 2-compact, second order bipolar fuzzy 2*- compact, second order bipolar fuzzy 3- compact are introduced by extending the definition of bipolar fuzzy compactness.

Relations between five second order bipolar fuzzy compactness:


Second order bipolar fuzzy 2* - compact
Second order bipolar fuzzy 1* - compact
Second order bipolar fuzzy 3 - compact
Second order bipolar fuzzy 2 - compact
Second order bipolar fuzzy 1 - compact

The relations between these concepts are analysed with special reference to R_1, R_2, R_3, R_4 and R_5. It is proved that all the five concepts of second order bipolar fuzzy compact spaces are preserved under second order bipolar fuzzy continuous function.

Chapter 6:

The sixth chapter is devoted to the study of second order bipolar fuzzy matrix and application of second order bipolar fuzzy TOPSIS method in Multi-Criteria Decision Making (MCDM) problem.

In 2019, M. Pal and Sanjib Mondal introduced the concept of bipolar fuzzy matrix. MCDM is a process to make an optimal choice that has the highest degree of achievement from a set of alternatives which are characterized in terms of multiple conflicting criteria. Hwang and Yoon (1981) devised the Technique for the Order of Preference by Similarity to Ideal Solution (TOPSIS) method, which is one of the most advantageous and successful approaches for resolving MCDM problems. In classical MCDM methods, the attribute values and weights are determined precisely. In 2000, Chen introduced the fuzzy version of TOPSIS method to deal with problems consisting of incomplete and vague information. In 2002, Chung and Chu presented fuzzy TOPSIS method under group decision for facility location selection problem. In 2014, Hadi et al. proposed the fuzzy inferior ratio method for multiple attribute decision making problems. In 2016, Dey et al. considered TOPSIS method for solving the decision making problem under bipolar neutrosophic fuzzy environment. In 2018, Alghamdi et al. studied multi-criteria decision-making methods in bipolar fuzzy environment.

In the first section, the concept of second order bipolar fuzzy matrix (SOBPFM) is introduced.

"A Second Order Bipolar Fuzzy Matrix (SOBPFM) is defined as an ordered pair $\left(\left(\widehat{A}_{bp}\right)_{ij}\right)_{p\times q} = \left(\left(\widehat{A}_{bp}{}^{+}, \widehat{A}_{bp}{}^{-}\right)_{ij}\right)_{p\times q}$ such that $\left(\widehat{A}_{bp}{}^{+}\right)_{ij}(x)(\alpha) \in [0,1]$ and $\left(\widehat{A}_{bp}{}^{-}\right)_{ij}(x)(\alpha) \in [-1,0]$, $\forall i, j,$ where $\alpha \in I \,\&\, x \in X$ and p, q represents the rows and columns of the matrix."

The transpose, identity and trace of a SOBPFM are defined and are denoted as $\left(\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{T}, \left(\hat{I}_{bp}\right)_{n} \,\&\, tr\left(\widehat{A}_{bp}\right)_{ij}\right)$ respectively.

Operations such as

(i) $\left(\left(\widehat{A}_{bp}\right)_{ij} + \left(\hat{B}_{bp}\right)_{ij}\right)_{p\times q}$

(ii) $\left(\left(\widehat{A}_{bp}\right)_{ij} * \left(\hat{B}_{bp}\right)_{ij}\right)_{p\times q}$

(iii) $\left(\left(\left(\widehat{A}_{bp}\right)_{ij}\right)^{c}\right)_{p\times q}$

Basic properties such as

$\left(\widehat{A}_{bp}\right)_{ij} * \left(\left(\widehat{B}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij}\right) = \left(\left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij}\right) * \left(\widehat{C}_{bp}\right)_{ij}$ (Associativity)

(iv) $\left(\widehat{A}_{bp}\right)_{ij} * \left(\left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{C}_{bp}\right)_{ij}\right) = \left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{B}_{bp}\right)_{ij} + \left(\widehat{A}_{bp}\right)_{ij} * \left(\widehat{C}_{bp}\right)_{ij}$ (Distributive law)

and some propositions are proved with examples.

In the second section, a working procedure for second order bipolar fuzzy TOPSIS (**T**echnique for the **O**rder of **P**reference by **S**imilarity to **I**deal **S**olution) method is presented and an application is provided in which an optimal solution to the MCDM problem for selecting a qualified applicant for the position of marketing manager has been found.

Conclusion:

The learning outcomes of the present work are given as follows:

Five distinct relations between first and second order bipolar fuzzy topological spaces are derived in chapter 2. Additionally, relations between second order bipolar fuzzy topological spaces and crisp topological spaces are discovered.

The third chapter contains verifications of the perseverance of second order bipolar fuzzy continuity and second order bipolar fuzzy product topology through the relations presented in the second chapter.

The perseverance of different versions of second order bipolar fuzzy Hausdorffness with respect to the relations introduced in the second chapter were verified in the fourth chapter.

In chapter 5, five relations between first and second order bipolar fuzzy gradation of openness and first order gradation of openness are established. It is proved that the relations are functorial with respect to gradation preserving maps. Results related to the second order bipolar fuzzy topologies induced by the second order bipolar fuzzy gradation of openness are obtained. Further, perseverance of second order bipolar fuzzy compactness about the relations were verified.

In the sixth chapter, properties of second order bipolar fuzzy matrix such as associative law and distributive law are proved. Also, by applying second order bipolar fuzzy TOPSIS method in a real life problem of choosing a suitable candidate for the job of marketing manager, an optimal decision has been given.

References:

[1] Alghamdi, M.A., Alshehri, N.O. and Akram, M. Alghamdi, M.A. and Alshehri, N.O. (2018), Multi-criteria decision-making methods in bipolar fuzzy environment. *Int. J. Fuzzy Syst*, 20, (6), 2057–2064.

[2] Azhagappan, M., Kamaraj, M. (2016), Notes on Bipolar Valued Fuzzy RW-closed and RW-open sets in Bipolar Valued Fuzzy Topological Spaces. Int. J. Math.Arch,7, 30-36.

[3] Chattopadhyay, K.C., Hazra, R.N. and Samanta, S.K. (1992), Gradation of Openness: Fuzzy Topology, *Fuzzy Sets and Systems*, **49**, 237 – 242.

[4] Chen, C.T. (2000), Extensions of the TOPSIS for group decision-making under fuzzy environment. *Fuzzy Sets System,* 114, 1–9. 15.

[5] Chu, T.C. (2002), Facility location selection using fuzzy TOPSIS under group decisions. Int. J. Uncertain. Fuzziness *Knowl.-Based Syst*, 10, 687–701.

[6] Dey, P.P., Pramanik, S., Giri, B.C. (2016), TOPSIS for solving multi-attribute decision making problems under bipolar neutrosophic environment. In New Trends in Neutrosophic Theory and Applications; *Pons Editions: Brussels, Belgium*, 65–77.

[7] Gantner, T.E., Steinlage, R.C. and Warren, R.H. (1978), Compactness in Fuzzy Topological Spaces, *Journal of Mathematical Analysis and Applications*, **62**, 547 - 562.

[8] Hadi-Vencheh, A., Mirjaberi, M. (2014), Fuzzy inferior ratio method for multiple attribute decision making problems. Inf. Sci, 277, 263–272.

[9] Hwang, C.L. and Yoon, K., (1981), Multi Attribute Decision Making: *Methods and Applications;* Springer: New York, NY, USA.

[10] Kalaichelvi, A. (2000), A Study of Second Order Fuzzy Structures, *Ph.D. Thesis*, Avinashilingan Institute for Home Science and Higher Education for Women, Coimbatore, Tamil Nadu, India.

[11] Kalaichelvi, A. (2007), Second Order Fuzzy Topological Spaces – I, *ActaCienciaIndica*, **33** (3), 819 – 826.

[12] Kalaichelvi, A. (2011), Second Order Fuzzy W – Hausdorff Spaces, *International Journal of Mathematical Sciences and Applications*, **1** (3), 1427 - 1431.

[13] Kalaichelvi, A. (2013), Second Order Fuzzy S – Hausdorff Spaces, *International Journal of Fuzzy Mathematical Archive*, **1**, 41 - 48.

[14] Katsaras, A.K. (1981), Ordered Fuzzy Topological Spaces, *Journal of Mathematical Analysis and Applications*, **84**, 44 – 58.

[15] Kim, J., Samanta, S. k., Lee, P. K., Hur, K. (2019), Bipolar Fuzzy Topological Spaces, *Annals of Fuzzy Mathematics and Informatics*, **17**, 205-229.

[16] Lowen, R. (1976), Fuzzy Topological Spaces and Compactness, *Journal of Mathematical Analysis and Applications*., **56**, 621-633.

[17] Muthuraji, T., Anitha, A. (2023), Recent Technology in Agriculture Using the Bipolar Fuzzy Matrices With the Aid of Score Function, Fifth IEEE International Conference on Electrical, Computer and Communication Technologies (ICECCT).

[18] Muthuraji, T., Anitha, A. (2023), Application of Bipolar Fuzzy Matrix in the Research of Cropson Agriculture, Indian Journal of Natural Science, **79**, 59188-59193.

[19] Muthuraji. T., Punitha Elizabeth, P. (2023), "Application of Bipolar Fuzzy Matrices in FloodDamage", Fifth IEEE International Conference on Electrical, Computer and Communication Technologies (ICECCT).

[20] Pal, M., Mondal, S., Bipolar Fuzzy Matrices, Soft Computing, 2019, **23(**20), 9885-9897.

[21] Srivastava, R., Lal, S.N. and Srivastava, A.K. (1981), Fuzzy Hausdorff Topological Spaces, *Journal of Mathematical Analysis and Applications*, **81**, 497 – 506.

[22] Zadeh, L.A. (1965), Fuzzy Sets, *Information and Control*, **8**(3), 338-353.

[23] Zadeh, L.A. (1975), The Concept of a Linguistic Variable and its Application to Approximate Reasoning – I, *Information Sciences*, **8**, 199 - 249.

[24] Zhang, W.R. (1994), Bipolar Fuzzy Sets and Relations : a Computational Framework for Cognitive Modelling and Multiagent Decision Analysis, Proc. of IEEE., 305-309.

Publications:

- Muthamizhselvi, S. and Vijayalakshmi, V. M. (2024), A Study on First and Second Order Bipolar Fuzzy Topological Spaces and Crisp Topological Spaces and Analyzing the Connections Between Them, *International Journal of Neutrosophic Science*, Vol.24, No.4, pg.133-150. (Scopus Indexed)

- Second Order Bipolar Fuzzy Matrix and its Application in Medical Diagnosis, *Communications on Applied Nonlinear Analysis*, Accepted.(Scopus Indexed)